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Analytical treatment of the one-dimensional Coulomb problem for the spinless Salpeter equation

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Abstract
Using the momentum space representation, we present an analytical treatment of the one-dimensional spinless Salpeter equation with a Coulomb interaction. The exact bound-state energy equation is determined. The results obtained are shown to agree very well with exact numerical calculations existing in the literature.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The spinless Salpeter equation (SSE) is a simple semi-relativistic version of the Schrödinger equation which incorporates the exact free-particle relativistic relation between energy and momentum. This equation is generally used when kinetic relativistic effects cannot be neglected. It is suitable for the description of scalar bosons as well as the spin-averaged spectra of bound states of fermions. Moreover, the spinless Salpeter equation appears as a standard approximation to the Bethe–Salpeter formalism which is the basic description of bound states within the framework of relativistic quantum field theory [1]. It can be deduced from the so-called Salpeter equation [2] by neglecting the spin degree of freedom and disregarding negative-energy solutions. The Salpeter equation, in turn, approximates the Bethe–Salpeter equation by assuming an instantaneous interaction [3]. The one-particle spinless Salpeter equation for stationary states has the form of an eigenvalue problem ($\hbar = c = 1$):

$$H|\psi_k\rangle = E_k|\psi_k\rangle, \qquad k = 1, 2, 3, \ldots,$$

with E_k being the energy eigenvalues corresponding to Hilbert-space eigenvectors $|\psi_k\rangle$, and the Hamiltonian H of the system being of the form

$$H = \sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{r}),$$

where $\sqrt{\mathbf{p}^2 + m^2}$ is the relativistic mass plus kinetic energy of the particle of mass m and momentum $\mathbf{p}$, and $V(\mathbf{r})$ is an arbitrary position-dependent static interaction potential. Due to the presence of the square root which renders the Hamiltonian non-local, the construction of analytical solutions to this equation is generally not a trivial task. Therefore, the investigation of this equation always implies either the use of approximating methods or the application of numerical approaches. The one-dimensional Coulomb potential is one among several problems which have been investigated on the basis of the spinless Salpeter equation [4–8, 10]. However, to our knowledge, an exact analytic expression of its energy eigenvalues is still absent from the literature (for a detailed history review of the spinless Salpeter Coulomb problem we refer the reader to [11]). In this paper, we fill this gap and present an exact analytical treatment of the one-dimensional spinless Salpeter Coulomb problem in the momentum space representation. Note that besides the interesting applications of the one-dimensional Coulomb potential in theoretical physics (such as the investigation of the mass spectra of mesons [12]), the solution of this problem may serve as a toy model which can be instructive for the analysis of the more important three-dimensional case.

Our paper is organized as follows. In section 2, we find the exact solution of the one-dimensional spinless Salpeter Coulomb problem in the momentum space. In section 3, we determine the quantization condition and obtain the exact energy spectrum. The resulting binding energy levels are then calculated and compared to those found in the literature by exact numerical treatment of the problem. The final section contains our conclusions.

2. One-dimensional spinless Salpeter Coulomb problem in the momentum space

In this section, we are interested in the solution of the one-dimensional eigenvalue equation,

$$[\sqrt{p^2 + m^2} + V(x)]|\psi\rangle = E|\psi\rangle, \tag{1}$$

with a Coulomb-type interaction,

$$V(x) = -\frac{\kappa}{x}, \qquad x \in \mathbf{R} \quad (\kappa > 0). \tag{2}$$

The same problem was considered in [10], but the configuration space was restricted to the positive half line:

$$\psi(x) = 0 \qquad \text{for} \quad x \leqslant 0. \tag{3}$$

This restriction may be interpreted as due to the presence of a 'hard-core' interaction potential effective for $x \leqslant 0$. Before starting our solution, let us review the main features of this work and make some comments on it. In [10], the author began by showing that the action of the kinetic energy operator $\sqrt{p^2 + m^2}$ on an element $\psi(x)$ of $L_2(\mathbf{R})$ can be expressed as

$$\sqrt{-d_x^2 + m^2}\psi(x) = \frac{1}{\pi}\left(-d_x^2 + m^2\right)\int_0^\infty \mathrm{d}y\, K_0(my)[\psi(x+y) + \psi(x-y)], \tag{4}$$

where $d_x^2 = \mathrm{d}^2/\mathrm{d}x^2$, and $K_0(x)$ is the modified Bessel function of order 0. He worked out this action for the special case of the function $x^n\,\mathrm{e}^{-\beta x}$ and put it under the form

$$\sqrt{-d_x^2 + m^2}\,x^n\,\mathrm{e}^{-\beta x} = \mathrm{e}^{-\beta x}\sum_{k=0}^{n} F_{k,n}(m,\beta)x^{n-k}, \tag{5}$$

with some coefficients $F_{k,n}(m,\beta)$ (see (17)–(20) in [10]). To take advantage of this result, the author tackled the problem in the configuration space where (1) takes the form

$$\sqrt{-d_x^2+m^2}\psi(x)=\left(E+\frac{\kappa}{x}\right)\psi(x), \tag{6}$$

and he supposed that the wavefunctions have the following expression:

$$\psi(x)\propto\sum_{k=0}^{n}\gamma_{k,n}\,x^k\,\mathrm{e}^{-\beta x} \qquad \text{for} \quad x>0 \quad \text{and} \quad n=1,2,\ldots, \tag{7}$$

$$\psi(x)=0 \qquad \text{for} \quad x\leqslant 0. \tag{8}$$

However, he made an error when he used result (5) in expressing the action of the operator $\sqrt{-d_x^2+m^2}$ on the wavefunction $\psi(x)$ in (6). Indeed, (5) was obtained by performing the integrations in (4) without taking into account the condition $\psi(x\leqslant 0)=0$. To be precise, in establishing expression (5) the author writes

$$\int_0^\infty \mathrm{d}y\,K_0(my)\psi(x-y)=\mathrm{e}^{-\beta x}\int_0^\infty \mathrm{d}y\,K_0(my)(x-y)^n\,\mathrm{e}^{\beta y}. \tag{9}$$

This equation would be correct if we considered the whole x-axis, but when calculating the action of $\sqrt{-d_x^2+m^2}$ on $\psi(x)$, the domain of integration is constrained by the condition $\psi(x\leqslant 0)=0$ so that the above integral reduces to

$$\int_0^\infty \mathrm{d}y\,K_0(my)\psi(x-y)=\mathrm{e}^{-\beta x}\int_0^x \mathrm{d}y\,K_0(my)(x-y)^n\,\mathrm{e}^{\beta y}. \tag{10}$$

This point was missed by the author. The obtained expressions for the energy spectrum and the wavefunctions are incorrect. The former was given as

$$E_n=\frac{m}{\sqrt{1+\frac{\kappa^2}{n^2}}}, \qquad n=1,2\ldots, \tag{11}$$

which is the same expression of the energy spectrum obtained in [4]. In [9], Lucha *et al* have shown that the above set of energy levels is in clear conflict with upper bounds on the energy eigenvalues derived using a variational procedure:

$$E_n\leqslant\sqrt{1-\frac{\kappa^2}{n^2}}, \qquad n=1,2\ldots. \tag{12}$$

Other arguments proving the non-validity of the results of [10] were also given.

Let us now return to our analysis. First, we note that potential (2) presents the same bound states as considered in [10]. This is due to the fact that the bound states are determined by the potential well of the right-half space (which is the same for the two types of interaction) and the boundary condition $\psi(x=0)=0$. In our case, this condition is a consequence of the infinite barrier that potential (2) possesses in the left-half space.

In order to avoid complications related to the non-locality of the square-root operator, we choose to work in the momentum space. The method we shall follow to solve problem (1) is similar to that used in [4]. Operating on both sides of (1) by x and replacing the position operator by its momentum representation,

$$x=\mathrm{i}\frac{\partial}{\partial p}, \tag{13}$$

we get the following first-order differential equation:

$$\frac{\partial}{\partial p}[(E-\sqrt{p^2+m^2})\psi(p)]-\mathrm{i}\kappa\psi(p)=0. \tag{14}$$

Next let

$$\varphi(p) = (E - \sqrt{p^2 + m^2})\psi(p), \tag{15}$$

the resulting equation for $\varphi(p)$ reads

$$\frac{\partial \varphi(p)}{\partial p} - \frac{\mathrm{i}\kappa}{E - \sqrt{p^2 + m^2}}\varphi(p) = 0. \tag{16}$$

The solution of this equation is

$$\varphi(p) = \lambda_E (p + \sqrt{p^2 + m^2})^{-\mathrm{i}\kappa} \left[\frac{\sqrt{m^2 - E^2} - \mathrm{i}(p + \sqrt{p^2 + m^2} - E)}{\sqrt{m^2 - E^2} + \mathrm{i}(p + \sqrt{p^2 + m^2} - E)} \right]^{\eta}, \tag{17}$$

with $\eta = \frac{\kappa E}{\sqrt{m^2 - E^2}}$, and λ_E is an energy-dependent constant which can be determined from the normalization condition of the momentum space wavefunction $\psi(p)$:

$$\int_{-\infty}^{+\infty} |\psi(p)|^2 \, \mathrm{d}p = 1. \tag{18}$$

Substituting $\psi(p)$ by its expression and performing the above integration, we obtain

$$\lambda_E = \sqrt{\frac{m^2 - E^2}{2}} \left[E + \frac{2m^2}{\sqrt{m^2 - E^2}} \arcsin\left(\sqrt{\frac{m + E}{2m}} \right) \right]^{-\frac{1}{2}}. \tag{19}$$

The final expression of the normalized momentum space wavefunction reads

$$\psi(p) = \lambda_E \frac{(p + \sqrt{p^2 + m^2})^{-i\kappa}}{E - \sqrt{p^2 + m^2}} \left[\frac{\sqrt{m^2 - E^2} - \mathrm{i}(p + \sqrt{p^2 + m^2} - E)}{\sqrt{m^2 - E^2} + \mathrm{i}(p + \sqrt{p^2 + m^2} - E)} \right]^{\eta}. \tag{20}$$

Note that choosing the quantization condition from the requirement of single-valuedness of the eigenfunction $\psi(p)$ in the complex p-plane by imposing $\eta = n$ leads to the same incorrect energy spectrum given by (2). Here it would be a mistake to write that single-valuedness of $\psi(p)$ is equivalent to the condition $\eta = n$. In doing so, we neglect the influence of the first term in (20) on the single-valuedness. That is to say that the constraint $\eta = n$ is not sufficient to guarantee single-valuedness of $\psi(p)$ so that it cannot be imposed as a legitimate quantization condition. This was not the situation in the non-relativistic case where the kinetic energy does not involve any square-root term. In the following section we shall discuss the correct quantization condition.

3. Energy spectrum

In the momentum space representation, the action of the operator $1/x$ can be expressed as [13]

$$\frac{1}{x}\psi(p) = -\mathrm{i}\int_{-\infty}^{p} \psi(q) \, \mathrm{d}q. \tag{21}$$

The wavefunction $\psi(p)$ given by expression (20) is a solution of (14) but it does not satisfy the initial eigenvalue problem (1) with the definition (21) for the action of the operator $1/x$. To overcome this difficulty, we redefine the action of the operator $1/x$ in the momentum space as

$$\frac{1}{x}\psi(p) = -\mathrm{i}\int_{-\infty}^{p} \psi(q) \, \mathrm{d}q + \gamma, \tag{22}$$

where γ is a constant to be determined from the requirement that the eigenfunction $\psi(p)$ given by (20) should be a solution of the eigenvalue equation (1). The constant γ is then obtained from the following equation:

$$(E - \sqrt{p^2 + m^2})\psi(p) - \mathrm{i}\kappa \int_{-\infty}^{p} \psi(q)\,\mathrm{d}q + \kappa\gamma = 0. \tag{23}$$

Substituting the eigenfunction $\psi(p)$ by its expression (20), we get

$$\gamma = -\frac{\lambda_E}{\kappa}\left(\frac{\sqrt{m^2 - E^2} + \mathrm{i}E}{\sqrt{m^2 - E^2} - \mathrm{i}E}\right)^{\eta} \exp[-\mathrm{i}\theta_E] \tag{24}$$

with

$$\theta_E = \kappa \int_0^{\infty} \frac{\mathrm{d}u}{E - \sqrt{u^2 + m^2}}. \tag{25}$$

Note that due to the indetermination of the limit,

$$\lim_{p\to-\infty} (p + \sqrt{p^2 + m^2})^{-\mathrm{i}\kappa}, \tag{26}$$

an explicit expression for γ cannot be obtained. However, we can content ourselves with the form (24) which is rather more appropriate for the following manipulations.

Next, the quantization condition is obtained by demanding that the Hamiltonian of the system,

$$H = \sqrt{p^2 + m^2} - \frac{\kappa}{x}, \tag{27}$$

should be a Hermitian operator on its eigenfunction $\psi(p)$ given by (20). Obviously, the operator $\sqrt{p^2 + m^2}$ satisfies this condition; then our requirement implies that the operator $1/x$ should be Hermitian on the set of eigenfunctions $\psi_E(p)$:

$$\left\langle \frac{1}{x}\psi_E | \psi_{E'} \right\rangle = \left\langle \psi_E | \frac{1}{x}\psi_{E'} \right\rangle. \tag{28}$$

Replacing the operator $1/x$ by its momentum representation (22), the last condition reduces to

$$\begin{aligned} \mathrm{i}\int_{-\infty}^{+\infty} \psi_{E'}(p) \int_{-\infty}^{p} \psi_E^*(q)\,\mathrm{d}q\,\mathrm{d}p + \gamma^*[\psi_E] \int_{-\infty}^{+\infty} \psi_{E'}(p)\,\mathrm{d}p \\ = -\mathrm{i}\int_{-\infty}^{+\infty} \psi_E^*(p) \int_{-\infty}^{p} \psi_{E'}(q)\,\mathrm{d}q\,\mathrm{d}p + \gamma[\psi_{E'}] \int_{-\infty}^{+\infty} \psi_E^*(p)\,\mathrm{d}p. \end{aligned} \tag{29}$$

Taking into account the following integral formula,

$$\int_{-\infty}^{+\infty} f(x) \int_{-\infty}^{x} g(y)\,\mathrm{d}y\,\mathrm{d}x = \int_{-\infty}^{+\infty} g(y) \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}y\,\mathrm{d}x - \int_{-\infty}^{+\infty} g(y) \int_{-\infty}^{y} f(x)\,\mathrm{d}x\,\mathrm{d}y, \tag{30}$$

together with the orthogonality relation of the wavefunctions and observing that the eigenfunction $\psi_E(p)$ can be expressed as

$$\psi_E(p) = -\frac{\mathrm{i}}{\kappa}\frac{\partial\varphi_E(p)}{\partial p}, \tag{31}$$

where the function $\varphi(p)$ given by (17) is rewritten in a more useful form as

$$\varphi_E(p) = \lambda_E \left(\frac{\sqrt{m^2 - E^2} + \mathrm{i}E}{\sqrt{m^2 - E^2} - \mathrm{i}E}\right)^{\eta} \exp\left[\mathrm{i}\kappa \int_0^{p} \frac{\mathrm{d}u}{E - \sqrt{u^2 + m^2}}\right], \tag{32}$$

condition (29) simplifies to

$$\sin[\theta_E - \theta_{E'}] = 0 \tag{33}$$

which means that

$$\theta_E - \theta_{E'} = n\pi, \tag{34}$$

where n is an integer. Each of the two integrals has a divergent part independent of the energy which then cancel each other. Hence the resulting energy equation reads

$$\frac{2\kappa E}{\sqrt{m^2 - E^2}} \arctan\left(\sqrt{\frac{m+E}{m-E}}\right) = n\pi + \delta, \tag{35}$$

with δ being an arbitrary constant. The value of δ can be deduced from the results of an exact numerical solution of the problem given by (1) and (2).

Note that the obtained indetermination in the energy equation can be related to the fact that the boundary condition $\tilde{\psi}(x = 0) = 0$ cannot be imposed. Indeed, since $\tilde{\psi}(x)$ is the Fourier transform of $\psi(p)$, this condition amounts to

$$\tilde{\psi}(x = 0) = \int_{-\infty}^{+\infty} \psi(p)\,\mathrm{d}p = 0 \tag{36}$$

with relation (31), this condition transforms as

$$[\varphi(p)]_{-\infty}^{+\infty} = 0, \tag{37}$$

which, in turn, yields

$$\sin\left[\kappa \int_0^{\infty} \frac{\mathrm{d}p}{E - \sqrt{p^2 + m^2}}\right] = 0. \tag{38}$$

The problem with the above equation is that, besides the term $(n\pi + \delta)$, the integral gives a divergent contribution:

$$\lim_{p\to\infty} \ln(p/m + \sqrt{1 + p^2/m^2}).$$

One may think to overcome this difficulty by introducing a momentum cutoff $\Lambda \gg m$ with the boundary condition $\psi(\Lambda) = 0$ (this regularization is equivalent to replacing the potential at short distances by an infinite repulsive barrier) so that (38) becomes

$$\sin[\kappa \ln(\Lambda/m + \sqrt{1 + \Lambda^2/m^2}) + n\pi + \delta] = 0 \tag{39}$$

which allows us to deduce

$$\delta = -\kappa \ln(\Lambda/m + \sqrt{1 + \Lambda^2/m^2}). \tag{40}$$

However, it is not clear, in that case, how one can choose the value of the momentum cutoff Λ.

In order to get an idea of the dependence of the energy levels on the value of the parameter δ, we have plotted in figure 1 the variations with values of the parameter δ of both the ground-state and the first excited-state energies for the spinless Salpeter Coulomb problem (in all the calculations of the paper we take κ to be the fine structure constant). Next, let us examine the non-relativistic limit of (35). The latter is obtained by setting $E = m + E_{\mathrm{nr}}$ (E_{nr} is the non-relativistic binding energy) and expanding expression (35) with the assumption that $E_{\mathrm{nr}} \ll m$. Rewriting (35) as

$$\frac{2\kappa E}{\sqrt{m^2 - E^2}}\left[\frac{\pi}{2} - \arctan\left(\sqrt{\frac{m-E}{m+E}}\right)\right] = n\pi + \delta \tag{41}$$

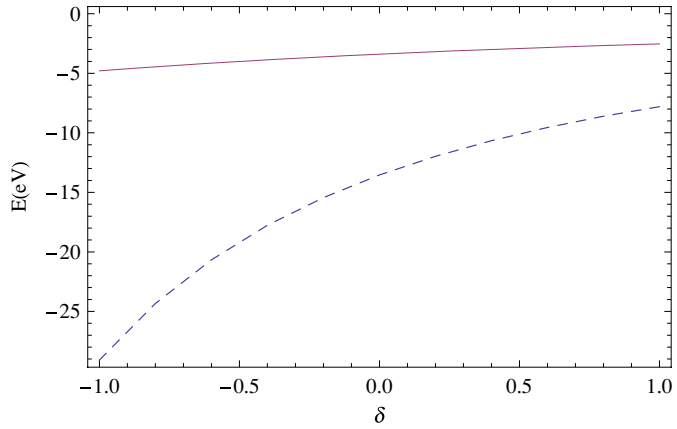

Figure 1. Plot of the energy eigenvalues of the Coulomb potential as a function of the parameter δ. Two energy levels are represented: the ground-state energy (dashed line) and the first excited-state energy (full line).

and expanding the above equation to leading terms in $E_{\rm nr}/m$, we obtain

$$\frac{\kappa\pi}{\sqrt{-2E_{\rm nr}/m}} - \kappa = n\pi + \delta. \tag{42}$$

The usual non-relativistic energy spectrum is recovered by setting $\delta = -\kappa$. However, this requirement cannot be used to fix δ in (35) since the same limit would be obtained if one replaced δ by a function of $\kappa : \delta(\kappa)$, satisfying the condition

$$\lim_{\kappa\to 0} \delta(\kappa)/\kappa = -1 \tag{43}$$

and expanded (42) to first order in κ.

In the following, we will compare our analytical finding with exact numerical calculations existing in the literature. We will use the results of [14] where exact numerical solutions of both the one-body spinless Salpeter equation for an electron moving in a central Coulomb potential (spinless Salpeter hydrogen-atom problem) and the two-body spinless Salpeter equation for two equal-mass electrons interacting via a central Coulomb potential (spinless positronium problem) are presented in the momentum space. In the two-body case, the comparison is made possible by the equivalence of the one-particle and the equal-mass two-particle equations [15]. Let us recall how the two equations can be equated. Given the following two relativistic Coulombic Hamiltonians for the one- and two-particle problems:

$$H^{(1)} = \sqrt{\mathbf{p}^2 + m^2} - \frac{\kappa}{r}, \qquad r = |\mathbf{x}| \tag{44}$$

and

$$H^{(2)} = 2\sqrt{\mathbf{P}^2 + M^2} - \frac{\alpha}{R}, \qquad R = |\mathbf{X}|, \tag{45}$$

respectively. Applying the canonical transformation,

$$\mathbf{p} = 2\mathbf{P}, \qquad \mathbf{x} = \frac{\mathbf{X}}{2}, \tag{46}$$

and identifying the mass and the Coulomb strength parameters as

$$m = 2M, \qquad \kappa = \frac{\alpha}{2}, \tag{47}$$

Table 1. Comparison of the binding energies obtained by exact analytical solution (Analytical) and exact numerical solution (Numerical) of the one-dimensional spinless Salpeter equation with a Coulomb interaction. Upper bounds calculated from (12) are also listed. A negative sign before the energy values has been omitted. All values are given in eV.

n	Analytical	Numerical	Upper bounds
1	13.606 595	13.606 595	13.605 873
2	3.401 5698	3.401 5706	3.401 4344
3	1.511 7904	1.511 7909	1.511 7458
4	0.850 3762	0.850 3765	0.850 3565
5	0.544 2384	0.544 2386	0.544 2280
6	0.377 9422	0.377 9423	0.377 9360
7	0.277 6712	0.277 6713	0.227 6673
8	0.212 5916	0.212 5918	0.212 5890
9	0.167 9734	0.167 9735	0.167 9715
10	0.136 0583	0.136 0585	0.136 0569

we end up with

$$H^{(1)} = H^{(2)}. \tag{48}$$

Note that the energy spectrum in the one-dimensional case is exactly identical to that corresponding to the three-dimensional case with vanishing orbital angular momentum. To obtain the energy levels from (35), in the two cases (one- and two-body problems) we choose to fix the value of the parameter δ by demanding that the ground-state energy of [14] should be a solution of (35) with $n = 1$. This condition gives

$$\delta = -0.007\,464\,1912 \qquad \text{and} \quad \delta = -0.003\,691\,0191 \tag{49}$$

for the one- and the two-particle problem, respectively. Then our energy eigenvalues are calculated by solving equation (35) numerically for different values of quantum number n where the parameters m, κ and δ are each time replaced by their corresponding values. The results of our calculations together with the comparison to those of [14] are presented in tables 1 and 2. The values of the binding energy upper bounds calculated from (12) are also reported in the same tables. As we can see, our calculations are in perfect agreement with numerical findings, and upper bounds (12) are well respected by the obtained sets of energy levels.

Let us make a remark concerning the check of the respect of energy upper bounds. The obtained energy equation (35) contains an unknown parameter δ which should naturally depend on the coupling constant κ. Unfortunately, this fact makes it impossible to check the respect of available energy upper bounds by the energy levels given by (35) for arbitrary values of the coupling constant κ. Therefore, the upper-bounds check was made only in the two cases where the parameter δ was initially fixed.

We turn now to the examination of the obtained momentum space wavefunctions. With condition (35), the expression of the norm of bound-state wavefunctions takes the form

$$|\psi_n(p)| = \sqrt{\frac{\left(m^2 - E_n^2\right)E_n^2}{2\left[E_n^2 + m^2(n\pi + \delta)/\kappa\right]}}\left(\sqrt{p^2 + m^2} - E_n\right)^{-1}, \qquad n = 1, 2, \ldots. \tag{50}$$

To get an insight into the effect of relativistic kinematic on the wavefunction, figure 2 compares norm (50) with the norm of the Schrödinger momentum space wavefunction for the ground state. We see that for low momentum values, the two curves fit together. Hence, relativistic

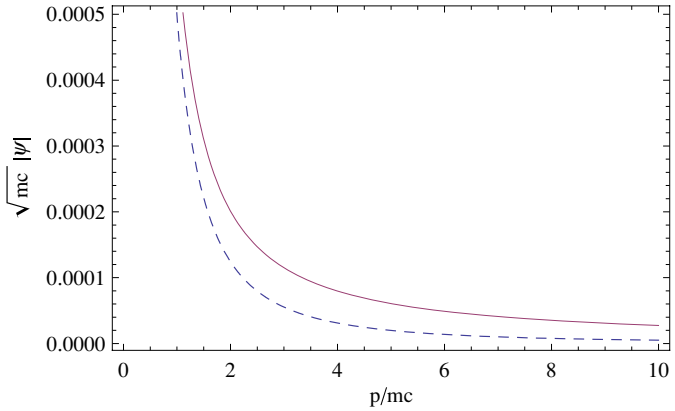

Figure 2. Comparison between ground-state momentum space wavefunctions corresponding to the Schrödinger equation (dashed line) and the spinless Salpeter equation (full line).

Table 2. Same as table 1 except for a spinless electron–positron bound state.

n	Analytical	Numerical	Upper bounds
1	6.802 9616	6.802 9616	6.802 8687
2	1.700 7306	1.700 7302	1.700 7129
3	0.755 8780	0.755 8777	0.755 8721
4	0.425 1806	0.425 1805	0.425 1780
5	0.272 1153	0.272 1152	0.272 1139
6	0.188 9688	0.188 9687	0.188 9680
7	0.138 8342	0.138 8341	0.138 8336
8	0.106 2949	0.106 2948	0.106 2945
9	0.083 9861	0.083 9860	0.083 9858
10	0.068 0288	0.068 0286	0.068 0285

corrections to the Schrödinger wavefunction are minor for small momentum. However, as the momentum value grows, the two curves separate. This is related to the different behaviours of the two wavefunctions at large momentum. Indeed, the Schrödinger wavefunction is equivalent to $1/p^2$ for $p \to \infty$; however, the spinless Salpeter wavefunction scales as $1/p$ when $p \to \infty$. The non-relativistic wavefunction decreases with the momentum faster than the relativistic wavefunction.

4. Conclusion

In summary, using functional analysis, we have solved exactly the one-dimensional spinless Salpeter equation for a Coulomb potential in the momentum space. Then we have derived the exact corresponding energy equation. To check the validity of the obtained results, we have compared the resulting binding energy levels to those previously found in the literature using exact numerical calculations for two different cases: the spinless Salpeter hydrogen-atom problem and the spinless Salpeter positronium problem. In the latter case, the comparison was made possible using the equivalence between the one-particle and the equal-mass two-particle equation. The agreement between the two results is almost perfect. We have also verified

that our sets of energy levels respect well energy upper bounds obtained using a variational method. Finally, we hope that the study presented in this paper will be helpful for the exact treatment of the still-unsolved three-dimensional spinless Salpeter Coulomb problem.

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